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A Method of Sampling Inspection

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This paper outlines some of the general considerations which must be taken into account in setting up any practical sampling inspection plan. An economical method of inspection is developed in detail for the case where the purpose of the inspection is to determine the acceptability of discrete lots of a product submitted by a producer. By employing probability theory, the method places a definite barrier in the path of material of defective quality and gives this protection to the consumer with a minimum of inspection expense.

ONE of the common questions in every day inspection work is, How Much Inspection? The answer must always be arrived at in the light of economy for only the least amount of inspection which will accomplish the purpose can be justified.

We wish here to consider the problem of setting up an economical inspection plan whose immediate purpose is the elimination of individual lots of product which are unsatisfactory in quality. By a lot of unsatisfactory quality is meant one that contains more than a specified proportion of defective pieces. This proportion is usually small and is based on economic considerations. The interest in inspections made for the elimination of such lots is shared by two parties,—the producer and the consumer. The consumer establishes certain requirements for the quality of delivered product. The producer so arranges his manufacturing processes and provides such an inspection routine as will insure the quality demanded. Our problem recognizes that the consumer runs some risk of receiving lots of unsatisfactory quality, if the quality of each lot is judged by the results of inspecting only a sample. The method of attack presumes the adoption of a risk whose magnitude is agreeable to the consumer and the selection of a particular inspection procedure which will involve a minimum of inspection expense on the part of the producer while guaranteeing the protection agreed upon.

Considering various possible concepts of a risk, the consumer would prefer to have one adopted that was worded as follows:

> "Not more than a specified proportion of the delivered lots shall be unsatisfactory in quality."

In other words, he would like to have the assurance that "the risk

of receiving a lot of unsatisfactory quality shall not exceed some definite figure." This risk involves two probabilities:

(*a*) that an unsatisfactory lot will be submitted for inspection.
(*b*) that the inspector will pass as satisfactory, an unsatisfactory lot submitted for inspection.

Without definite information regarding probable variations in the producer's performance and without an absolute assurance that this performance will remain consistently the same, the use of probability (*a*) in stating the risk to the consumer might be misleading. This probability will therefore receive no further consideration in this paper so far as the definition of the risk to the consumer is involved.

For probability (*b*) a reasonably low value for an upper limit can be given without any knowledge of the producer's performance. This upper limit (as defined below) has been taken as the starting point for the inspection method presented in this paper. The concept of risk which has been adopted gives the consumer unconditional assurance that his "chance of getting any unsatisfactory lot *submitted for inspection* shall not exceed some definite magnitude." This magnitude is the Consumer's Risk used.

Our problem may hence be stated as follows:

> Given a product of a specified type of apparatus or material coming from a producer in discrete lots, what inspection plan will involve a minimum of inspection expense, and at the same time insure that under no conditions will more than a specified proportion of the unsatisfactory lots submitted for inspection be passed for delivery to consumers?

This problem is economic in character and its solution involves the use of probability theory for establishing the height of the barrier to be placed in the path of unsatisfactory material.

Our attention will be directed particularly to the inspection of material which is produced more or less continuously on a quantity basis as distinguished from intermittent production in relatively small amounts. Under these conditions the producer is able to secure a continuing record of performance and to set up an inspection program which takes advantage of current quality trends.

General Considerations in Setting Up an Inspection Method

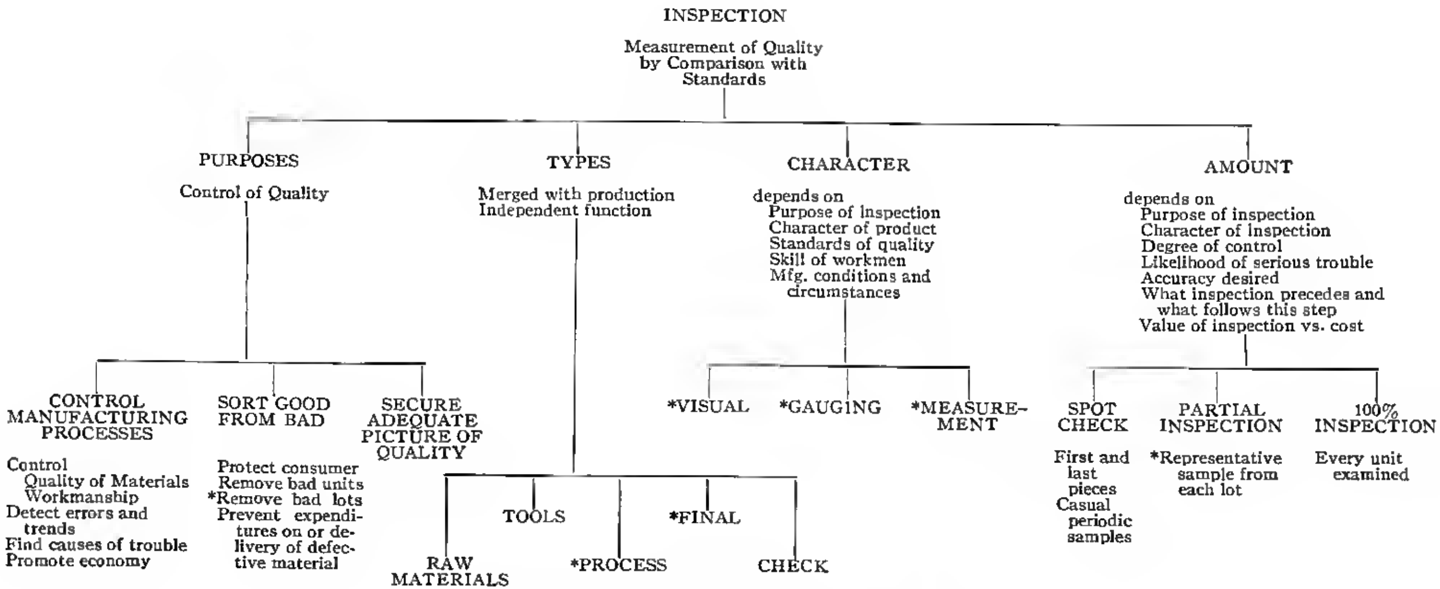
The broad purpose of inspection is to control quality by critical examinations at strategic points in the production process. Raw materials must be inspected. Some of the rough and finished parts must be inspected. In the manufacture of even the simpler kinds of merchandise, inspections dot the chart of progress from raw materials

to the finished product. The distribution of inspection activities throughout any process must be so ordered that the net cost of production will be consistent with the quality demanded by the customer. To determine whether an inspection should be made, or how much should be made at any one of the formative stages, is a major problem involving questions of both quality and economy.

One hundred per cent inspection is often uneconomical at a point in the production process where inspection is clearly warranted, particularly when preceded or followed by other inspections, inasmuch as the cost of more inspection at that point may not be reflected by a corresponding increase in the value of the finished product. In special cases, for example where inspection is destructive, 100 per cent inspection may be totally impracticable. Sampling inspections are often best from the standpoint of both the producer and the consumer when the value of quality and the cost of quality are weighed in the balance.

To arrive at an answer to the question, How Much Inspection, it is first essential to define clearly just what the inspection is intended to accomplish and to weigh all of the important factors both preceding and following the inspection in question which have a direct influence on the quality of the finished product and which as a whole determine how large a part this inspection step must play in controlling quality. Should it serve as an agency for making sure that the product at this stage conforms 100 per cent with the requirements for the features inspected? If so, 100 per cent inspection is required. Or should it serve to make reasonably certain that the quality passing to the next stage is such that no extraordinary effort would have to be expended on defective material? If so, sampling inspection may be employed. Ahead of all else, decisions are needed as to the specific requirements that must be satisfied by the inspection plan itself. This part of the problem is a practical one—one which must be approached in the light of experience, knowledge of conditions and the statistics of past performance. Once the basic requirements of the plan are agreed upon, probability theory can assist in formulating the details which will accomplish the desired results. It is important to hold in mind that statistical methods are aids to engineering judgment and not a substitute for it.

An attempt has been made in Fig. 1 to show schematically some of the outstanding general considerations which must be taken into account in establishing a proper setting for any problem that seeks to determine the economical amount of inspection. Inspections vary widely in purpose, type and character. While their broad purpose is



*Conditions involved in the inspection method of this paper.

Fig. 1—Chart showing some of the factors to be considered in establishing an inspection plan.

to control quality, the immediate objects of individual inspection steps differ. For example, the object of one step may be to secure information which will assist directly in controlling the manufacturing process by detecting errors or trends in performance which would become troublesome if allowed to persist unchecked. In other places the immediate object may be to determine the acceptability of definite quantities of product or to provide a screen for sorting the bad pieces or the bad lots from the good ones. Materials, parts in process and finished units are scrutinized with these objects in view. Depending on circumstances, the character and completeness of inspections vary from visual examination of small samples to careful measurement or testing of each piece.

Conditions Under Which the Present Method Applies

A large amount of industrial inspection work consists in comparing individual pieces with a standard—such as gauging a dimension or measuring an electrical property—to determine whether the pieces do or do not conform with the requirements given in specifications. This is often referred to as inspection by the "method of attributes." Consideration will be directed here to the case where non-destructive sampling inspection of this kind is conducted on discrete lots of product for the purpose of determining their acceptability.

From the standpoint of sampling theory, one of the general requirements is that each lot should be composed of pieces which were produced under the same essential conditions. Practically, this means that an attempt should be made to avoid grouping together batches of material, which, due to manufacturing conditions or methods, are apt to differ in quality. It is presumed, of course, that the sample drawn from any lot will be a random sample so that it may fairly represent the quality of the entire lot.

Summing up the general conditions for which a solution is sought, we assume

(1) The purpose of the inspection is to determine the acceptability of individual lots submitted for inspection, i.e., sorting good lots from bad.
(2) The inspection is made by the "method of attributes" to determine conformance with a particular requirement, i.e., each piece does or does not meet the limits specified.
(3) A lot is homogeneous in quality and the sample from it is a random sample.

The starred items in Fig. 1 indicate the set of conditions involved in our problem.

Protection and Economy Features of the Method

The adoption of sampling inspection at any stage of manufacture carries with it the premise that the product emerging from this point does not have to conform 100 per cent with specification requirements. It is often more economical, all things considered, to allow a small percentage of defective pieces to pass on to subsequent assembly stages or inspections for later rejection than to bear the expense of a 100 per cent inspection. Under these conditions, the status of the inspection can be clarified by establishing a definite tolerance for defects for the lots submitted to the inspector for acceptance. This may be specified as an allowable percentage defective, a figure which may be considered as the border line of distinction between a satisfactory lot and an unsatisfactory one. Thus, if the percentage defective is greater than this "tolerance per cent defective," the lot is unsatisfactory and should be rejected. We say "should be" rejected but this cannot be accomplished with absolute certainty if only a sample is examined. Sampling inspection involves taking chances since the exact quality of a lot is not known when only a part is inspected. According to the laws of chance, a sample will occasionally give favorable indications for bad lots which will result in passing them for delivery to consumers.

The first requirement for the method will therefore be in the form of a definite insurance against passing any unsatisfactory lot that is submitted for inspection.

The second requirement that will be imposed is that the inspection expense be a minimum, subject to the degree of protection afforded by the first requirement.

For the first requirement, there must be specified at the outset a value for the tolerance per cent defective as well as a limit to the chance of accepting any submitted lot of unsatisfactory quality. The latter has, for convenience, been termed the Consumer's Risk and is defined, numerically, as the probability of passing any lot submitted for inspection which contains the tolerance number of defects.

As will be shown further on, the first requirement can be satisfied with a large number of different combinations of sample sizes and acceptance criteria. To satisfy the second requirement, it is necessary then to determine the expected amount of inspection for a variety of inspection plans, determine the cost of examining or testing, add the costs other than those incurred in the simple process of examining samples, and choose among these plans that which involves a minimum of inspection expense.

There are, of course, a number of possible general methods of

inspection procedure, such as single sampling, double sampling, multiple sampling, etc., which allow the examination of only one sample, of two samples, or of more than two samples before a prescribed disposition of the entire lot is made. For each of these general methods, different combinations of sample sizes and acceptance criteria can be found which will satisfy the first requirement. We now prescribe that any lot which fails to pass the sampling requirements shall be completely inspected. Under this condition, one of the above mentioned combinations will give a lesser amount of inspection than the rest. Since a major cost item is that associated with the *amount* of inspection, we will carry through in detail the problem of finding the combination which will result in the *minimum* amount of inspection for one simple general method of inspection.

Single Sampling Method of Inspection

Attention is now directed to what is termed the Single Sampling method of inspection, which involves the following procedure:

(*a*) Inspect a sample.
(*b*) If the acceptance number for the sample is not exceeded, accept the lot.
(*c*) If the acceptance number is exceeded, inspect the remainder of the lot.

The term "Acceptance Number" is introduced to designate the allowable number of defects in the sample.

For this procedure, the first requirement reduces the problem to one which can be solved readily by determining probabilities associated with sampling from a finite lot containing the tolerance number of defects. For any sample size, there is a definite probability of finding no defects, of finding exactly one defect, exactly two defects, etc. If, under the above conditions, the acceptance number were 1, for example, there is one value of sample size such that the probability of finding one or less defects is equal to the value of the Consumer's Risk specified. Since a lot will be accepted if the observed number of defects does not exceed the acceptance number, the probability of finding one or less defects in a sample selected from a lot of tolerance quality is the risk of accepting any lot of tolerance quality submitted to the inspector. It follows that the risk of accepting a lot of worse-than-tolerance quality is less than the Consumer's Risk just defined. If the producer gets into trouble and begins to submit lots of unsatisfactory quality, the consumer has the assurance that his chance of getting them will not exceed this figure. In fact, the worse the quality, the less will be their chance of passing without a detailed

inspection. Thus the amount of inspection is automatically increased as quality degenerates.

For every acceptance number, such as 0, 1, 2, etc., there is an unique size of sample which will satisfy the specified values of tolerance per cent defective and Consumer's Risk. We thus have many pairs of values of sample size and acceptance number from which to choose.

The second requirement dictates which pair shall be chosen. We will select that pair which involves the least amount of inspection for product of *expected* quality. In industry, the quality emerging from any process tends to settle down to some level which may be expected more or less regularly day by day. If this level could be maintained quite constant, if the variations in quality were no larger than the variations that could be attributed to chance, then inspection could often be safely dispensed with. But practically, while such a level may be adhered to most of the time, instances of man failure or machine failure are bound to arise spasmodically and as a result the quality of the output may gradually or suddenly become unsatisfactory. The method of solution takes into consideration this usual or expected quality and requires an estimate of the expected quality under normal conditions. A satisfactory estimate of this can usually be obtained by reviewing data for a past period during which normal conditions existed and by utilizing such other pertinent information as bears on manufacturing performance under present or anticipated conditions. This expected value is defined as the *process average* to be used in the solution. Thus the method under discussion will assure the producer of a minimum of inspection expense so long as he holds to his expected performance. If he gets into trouble and the quality becomes poorer than normally expected, the method automatically increases the inspection by an amount which varies with the degree of quality degeneration. This reacts on the producer directly by increasing his inspection expense and serves as an incentive to the elimination of the causes of trouble. The producer's expected performance is thus made use of in a way that affects the economy of the producer's inspection work but is not used to color or affect the magnitude of the Consumer's Risk.

The amount of inspection, that will be done in the long run for uniform product [1] of process average quality is made up of two parts:

(1) The number of pieces inspected in the samples.
(2) The number of pieces inspected in the remainder of those lots which fail to be accepted when a sample is examined.

[1] By "uniform product" is meant one produced under a constant system of chance causes, giving rise to a quality which is a chance variable. In the present paper, this chance variable is assumed to be the Point Binomial.

But what proportion of the lots will fail to be accepted on the basis of the sampling results? Here is where probability theory comes in again. There will be a definite probability of exceeding the acceptance number in samples drawn from material of process average quality. Since we are interested in the amount of inspection *in the long run*, the sample at this stage of the problem may be regarded as drawn from a very large (mathematically infinite) quantity of homogeneous product whose percentage defective is equal to the process average per cent defective. Thus, for example, with an acceptance number of 1, the average [2] number of pieces inspected *per lot* as a result of *extended* inspections is equal to the number of pieces in the remainder of a lot multiplied by the probability of finding more than one defect in a sample drawn from an infinite quantity of material of this quality. This value plus the number of pieces inspected in the sample gives the average amount of inspection per lot for an acceptance number of 1. Similar results are found for all other acceptance numbers and the desired solution is obtained by choosing that acceptance number for which the average amount of inspection per lot is a minimum.

The plan thus provides the inspector with a definite routine to follow, such that his inspection effort will be a minimum under normal conditions.

Charts for Single Sampling

For any specified value of Consumer's Risk, charts may readily be constructed to give the acceptance number, the sample size, and the average number of pieces inspected per lot for the conditions outlined above. To illustrate the general character of these charts, Figs. 2, 3 and 4 are presented for a Consumer's Risk value of 10 per cent.

In the appendix it is shown that the acceptance number which satisfies the condition of minimum inspection is dependent on two factors, (1), the tolerance number of defects for a lot, and (2), the ratio of the process average to the tolerance for defects. Fig. 2 based on this relationship defines *zones* of acceptance numbers for which the inspection is a minimum.

Fig. 3 gives curves for finding the sample size. The mathematical basis for these curves is likewise given in the appendix. For a given tolerance number of defects and the acceptance number found from Fig. 2, the value of tolerance times sample size is determined. This quantity divided by the tolerance gives the sample size. The curves shown are based on an approximation which is satisfactory for practical use when the tolerance for defects does not exceed 10 per cent and the sample size is not extremely small.

[2] It is to be noted that wherever "average" appears in the paper, "expected" value, in the rigorous probability sense, is meant.

Fig. 4 gives curves which enable one to determine the minimum average number of pieces inspected per lot for uniform product of process average quality. For a given tolerance number of defects and a given ratio of process average to tolerance, a value of tolerance times minimum average number inspected per lot is determined. This when divided by the tolerance gives the desired value of minimum average number inspected per lot.

Illustrative Example

Suppose that lots of 1,000 pieces each are inspected for a characteristic having a specified tolerance of 5 per cent defective and that the process average quality of submitted lots is 1 per cent defective. If it is desired to have a risk of 10 per cent of accepting a 5 per cent defective lot, what single sampling plan should be followed by the inspector to give a minimum amount of inspection, and how much inspection per lot will be required on the average?

Referring to Fig. 2, for "Tolerance Number of Defects" ($.05 \times 1{,}000$) $= 50$ and "Ratio of Process Average to Tolerance" ($.01/.05$) $= .20$, we find the acceptance number $= 3$.

Referring to Fig. 3, for "Tolerance Number of Defects" $= 50$ and "Acceptance Number" $= 3$, we find "Tolerance Times Sample Size" $= 6.5$. Dividing by the tolerance (expressed as a fraction defective) $= .05$, gives a sample size of 130.

Referring to Fig. 4, for "Tolerance Number of Defects" $= 50$ and "Ratio of Process Average to Tolerance" $= .20$, we find by interpolation that the "Tolerance Times Minimum Average Number Inspected" $= 8.2$. Dividing by the tolerance, .05, gives an average number inspected per lot of 164.

This solution has thus been obtained by the initial specification of first, the tolerance per cent defective for a single lot and a value for the Consumer's Risk, these two factors being combined to give a definite measure of protection against passing faulty material, and second, a minimum amount of inspection for product of process average quality.

These two requirements control the exact details of inspection procedure and must be initially chosen on the basis of practical considerations and circumstances. The specification of these factors lends definiteness to the problem of inspection and provides a rationalized basis of procedure which can be depended on to give the desired degree of protection. Obviously, any value of Consumer's Risk may be chosen according to circumstances. The value which is proper in any case is dependent on the conditions associated with the product

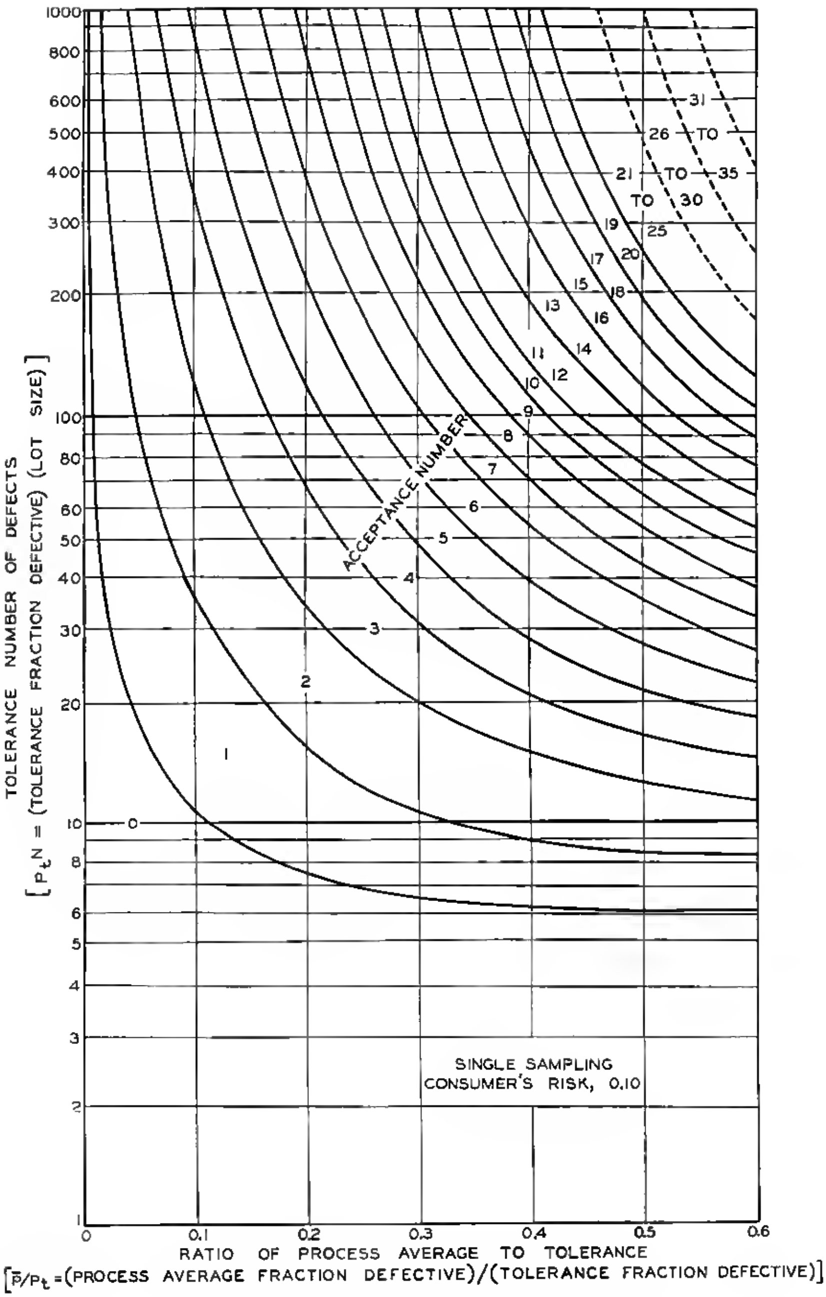


Fig. 2—Chart for finding acceptance number.

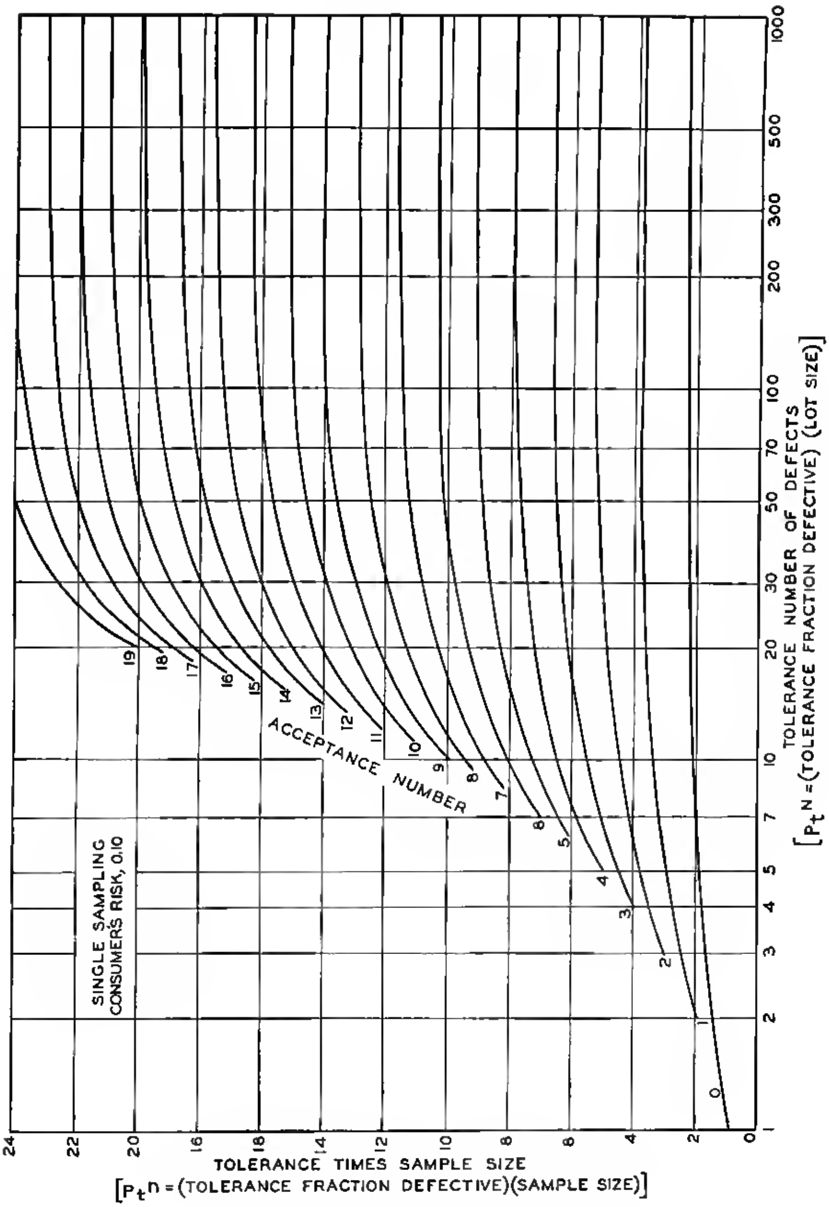


Fig. 3—Curves for finding size of sample.

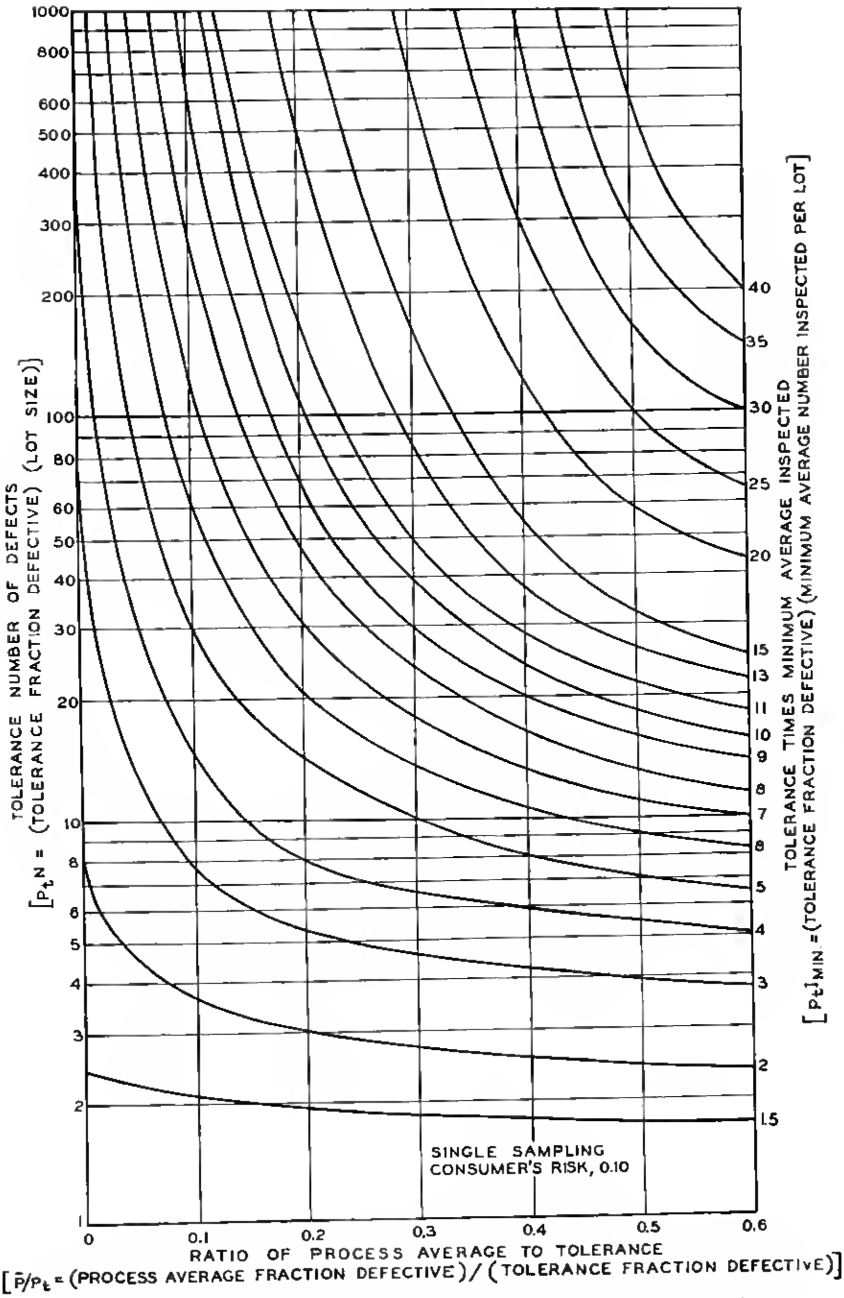


Fig. 4—Curves for finding the minimum amount of inspection per lot.

inspected, such as the degree of control exerted by the producing agencies, or the protective measures which precede or follow the inspection step in question. For any Consumer's Risk value, sets of charts similar to Figs. 2, 3, and 4 may be constructed.

Other Methods of Inspection

The above detailed discussion has been limited to one simple method of inspection, Single Sampling, in order to show how certain concepts and principles may be applied with the aid of sampling theory. The same principles are readily extended to a plan of double sampling or of multiple sampling in which cases a second sample may be examined if the first fails or a third, fourth, etc. examined if the preceding samples fail, before resorting to a detailed inspection of the remainder of a lot. As a matter of fact, for given values of tolerance and risk, the minimum average amount of inspection per lot will be somewhat less for plans which permit the examination of more than one sample before detailing, but when consideration is given to the costs associated with interruption of work, extraction of additional random samples, inconveniences or difficulties in handling the routine called for, etc., it has not been found economical in general to examine more than two samples from any lot.

It may be well to point out that other basically different requirements may be chosen for setting up economical sampling inspection plans. For example a satisfactory method has been devised to meet the following requirements:

(1) A limiting value to the average per cent defective after inspection.
(2) A minimum amount of inspection for product of process average quality.

This method has been found of value in continuous production where the inspection is intended to serve as a partial screen for defective units. It differs from that described above in that it provides a fixed limit to the *average* quality of product after inspection rather than a limit to the quality of each individual lot.

The solution of such problems, which employ probability theory as an aid, always demands a concise statement of the conditions and the specification of numerical requirements which the inspection must satisfy.

MATHEMATICAL APPENDIX

The problem considered is to minimize the average number of pieces inspected per lot in Single Sampling Inspection. The method and equations developed below may be extended to problems involving

2, 3, 4, etc., samples. The method of extension is somewhat complicated although the procedure is identical in nature. For example, in Minimum Double Sampling, first and second acceptance numbers and corresponding first and second sample sizes must be found, and at the same time the total Consumer's Risk must be properly divided between the two samples. We will restrict our attention to the problem of Minimum Single Sampling.

The first seven variables defined below in Table I enter into the two equations needed for the solution of the problem, the first three being fixed by the requirement of the method that a definite protection be provided against accepting faulty material, the fourth being fixed by the requirement that the average amount of inspection shall be a minimum for uniform product of process average quality. Therefore, the three unknown variables are c, n, and I. The five variables N, n, I, $\bar{p}$, and p_t are replaced in the solution by four variables which have been obtained from the original variables by combining p_t with the other four, viz. $M = p_tN$, $a = p_tn$, $z = p_tI$, and $k = \bar{p}/p_t$. Since M, P, and k are specified by the method, the unknown variables are c, a, and z. Two tables showing respectively the notation [3] and the disposition of the variables are presented below.

TABLE I

NOMENCLATURE

N = number of pieces in lot,
P = Consumer's Risk, the probability of accepting a submitted lot of tolerance quality,
p_t = tolerance fraction defective,
$\bar{p}$ = process average (expected) fraction defective,
c = acceptance number, the maximum allowable number of defective pieces in sample,
n = number of pieces in sample,
I = average (expected) number of pieces inspected per lot,
$M = p_tN$ = number of defective pieces in lot of tolerance quality,
$a = p_tn$ = expected number of defective pieces in sample drawn from lot of tolerance quality,
$z = p_tI$ = product of tolerance and average (expected) number of pieces inspected per lot,
$k = \bar{p}/p_t$ = ratio of process average to tolerance,
m = number of defects found in sample,
$C_n^N = \dfrac{N!}{(N-n)!n!}$ = number of combinations of N things taken n at a time.

The solution of the problem requires the consideration of the

[3] The symbols $\bar{p}$ and p_t, as used in this problem, are assumed true parameters of the universe sampled and according to the notation adopted by these Laboratories should be primed, i.e. $\bar{p}'$ and p_t'. For the sake of simplicity here the prime notation has been omitted in the equations. Ref. W. A. Shewhart, "Quality Control," *Bell System Technical Journal*, Vol. VI, p. 723, footnote 3, October, 1927.

following two equations:

$$z = f_1(M, a, c, k), \quad (1)$$

$$P = f_2(M, a, c), \quad (2)$$

where f_1 and f_2 represent symbolic functions wh.ch are to be determined later.

We wish to find a pair of values (c, a) which will make z a minimum,

TABLE II

Disposition of Variables

Initial Variables Involved	Initial Fixed Variables	Initial Unknown Variables	Variables Used in Method	Fixed Variables of Method	Unknown Variables of Method
N	N		$M = p_t N$	M	
P	P		P	P	
p_t	p_t		$k = \bar{p}/p_t$	k	
$\bar{p}$	$\bar{p}$		c		c
c		c	$a = p_t n$		a
n		n	$z = p_t I$		z
I		I			

subject to the condition that this pair (c, a) satisfies equation (2). Hence, due to the discreteness of c, pairs (c, a) satisfying (2) are substituted in (1) until a minimum value of z is found. Thus, for $P = .10$ and for given values of M and k, we read c from Fig. 2, a for this value of c from Fig. 3, and the minimum value of z from Fig. 4.

Basis of Fig. 2 Giving Minimum Acceptance Numbers

In determining the function f_1 involved in equation (1), the average number of pieces inspected per lot, I, is treated as the dependent variable. Since a sample is always taken, n pieces will be inspected from every lot submitted. The number of times that the remainder of the lot $(N - n)$ will be inspected on the average is determined from the expression giving the probability that more than c defective pieces will be found in n. The sample is assumed to be drawn from a product of which a fraction, $\bar{p}$, is defective. The probability that c or less defective pieces will be found in n pieces selected at random from a product containing $\bar{p}$ fraction defective pieces is given by the sum of the first $c + 1$ terms of the Point Binomial $[(1 - \bar{p}) + \bar{p}]^n$. Hence the average number of pieces inspected per lot is determined from the relation,

$$I = n + (N - n)\left[1 - \sum_{m=0}^{m=c} C_m^n (1 - \bar{p})^{n-m}\bar{p}^m\right].$$

For the condition, $\bar{p} < .10$, which is usual in practice, it has been found satisfactory to replace the Point Binomial by the Poisson Exponential.[4] By multiplying both sides of the equation by p_t we obtain z in the form,

$$z = M - (M - a)\sum_{m=0}^{m=c} \frac{(ka)^m e^{-ka}}{m!}, \tag{1'}$$

which is the function f_1 desired.

To obtain f_2, we state the probability of finding c or less defects in a sample n taken from a lot N containing $M = p_t N$ defective pieces. This is given by the equation,

$$P = \frac{1}{C_n^N}\sum_{m=0}^{m=c} C_{n-m}^{N-M} C_m^M.$$

But this equation is too difficult to handle in general computations on a large scale. When $p_t < .10$ and n is sufficiently large, a satisfactory approximation to the above equation may be developed from the first $c + 1$ terms of the Point Binomial, that is

$$P = \sum_{m=0}^{m=c} C_m^n \left(1 - \frac{M}{N}\right)^{n-m}\left(\frac{M}{N}\right)^m, \qquad \text{since} \qquad p_t = \frac{M}{N}.$$

An even better approximation is obtained by interchanging[5] n and M in the latter equation giving the expression,

$$P = \sum_{m=0}^{m=c} C_n^N \left(1 - \frac{n}{N}\right)^{M-m}\left(\frac{n}{N}\right)^m.$$

Since $\frac{a}{M} \equiv \frac{n}{N}$, we obtain the final form,

$$P = \sum_{m=0}^{m=c} C_m^M \left(1 - \frac{a}{M}\right)^{M-m}\left(\frac{a}{M}\right)^m, \tag{2'}$$

which is the function f_2 desired.

Now that we have f_1 and f_2 as expressed in equations (1′) and (2′) we must explain how Fig. 2 was obtained. When $P = .10$, for any pair (M, k), a particular pair (c, a) was found which made z a minimum. The acceptance number c may assume only discrete values since any

[4] G. A. Campbell, "Probability Curves Showing Poisson's Exponential Summation," *Bell System Technical Journal*, Vol. II, pp. 95–113, January, 1923.

[5] Paul P. Coggins, "Some General Results of Elementary Sampling Theory for Engineering Use," *Bell System Technical Journal*, Vol. VII, p. 44, Equation (11), January, 1928.

piece must be considered either as defective or non-defective. Hence minimum values of z ($z_{\text{min.}}$) will be found for many pairs (M, k) for the same value of c. From this it is evident that on an M, k plane there exist zones in which the acceptance numbers are identical. To find the boundary lines of these zones it was noted that for certain pairs (M, k) two pairs of (c, a) exist, giving the same minimum value for z. These values of c were found to differ by 1 in all such cases. Designating in general two such adjacent acceptance numbers as c and $c + 1$ and corresponding values of a which satisfy the Consumer's Risk as a_c and a_{c+1}, we may obtain these boundary curves from the equation,

$$(M - a_c) \sum_{m=0}^{m=c} \frac{(ka_c)^m e^{-ka_c}}{m!} = (M - a_{c+1}) \sum_{m=0}^{m=c+1} \frac{(ka_{c+1})^m e^{-ka_{c+1}}}{m!}.$$

In using the above equation to determine these boundary curves for Fig. 2, the following steps were taken:

(1) Assume values for c and $c + 1$.
(2) Determine a_c and a_{c+1} for a given value of P assuming N to be infinite.
(3) For any given value for k, solve the linear equation in M obtained by substituting the assumed values in the above equation.
(4) Using the value of M thus found, determine the exact values of a_c and a_{c+1} from equation (2′) for $P = .10$ (Fig. 3).
(5) Using the same value of k, again solve the linear equation in M substituting the values of a_c and a_{c+1} obtained from step (4).
(6) If the values a_c and a_{c+1} obtained in step (4) satisfy the value of M thus found, the values of M and k define a point on the boundary curve between two adjacent acceptance numbers. If these values of a do not satisfy the value of M thus determined, steps (4) and (5) may be repeated until the limiting conditions are satisfied.

Basis of Fig. 3 Giving n for Any c

For given values of the Consumer's Risk P and acceptance number c, the sample size n may be obtained from equation (2′) since $a = p_t n$. For the case $P = .10$ values of a are presented in Fig. 3 for selected ranges of M and c.

Basis of Fig. 4 Giving the Minimum Average Number of Pieces Inspected Per Lot

The curves in Fig. 4 represent specific values of $z_{\text{min.}}$ on an M, k plane for $P = .10$. Each curve was obtained by substituting given

values of $z_{\text{min.}}$, k, and (c, a) in equation (1′) and solving for M. If, for this value of M thus found, the selected values of c and a coincide with those read respectively from Figs. 2 and 3, a point was established for the given value of $z_{\text{min.}}$ on the M, k plane. If not, sufficient trials were made until the condition given by Figs. 2 and 3 were met. The curves for $z_{\text{min.}}$ were thus determined. To obtain $I_{\text{min.}}$ it was only necessary to use the relation $I_{\text{min.}} = \frac{z_{\text{min.}}}{p_t}$.